

Chap 8 Appendix A Addendum

Student Friendly QFT Volume 2

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1 Background

There are two bases we want to keep straight in our minds.

1.1 The Usual Minkowski Basis

The first of these is the usual spacetime basis for vector entities in flat spacetime. This is the Minkowski metric basis, where each orthogonal direction in space has a unit basis vector and the time direction also has a unit basis vector. These vectors can be represented as

$$\mathbf{i}_0, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, \quad (1)$$

where we have 4D coordinate axes of x^0 - x^1 - x^2 - x^3 .

If we want to express each of these in terms of the entire basis, we have

$$\mathbf{i}_0 = i_0^\mu = (1, 0, 0, 0) \quad \mathbf{i}_1 = i_1^\mu = (0, 1, 0, 0) \quad \mathbf{i}_2 = i_2^\mu = (0, 0, 1, 0) \quad \mathbf{i}_3 = i_3^\mu = (0, 0, 0, 1), \quad (2)$$

which is actually rather trivial (though I hate to use that word.) Any vector field, such as the Z field, can be expressed in terms of its four components in that basis (and that is what we usually do).

$$Z^\mu = Z^0 \mathbf{i}_0 + Z^1 \mathbf{i}_1 + Z^2 \mathbf{i}_2 + Z^3 \mathbf{i}_3 = Z^0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + Z^1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + Z^2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + Z^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{bmatrix}. \quad (3)$$

1.2 Polarization Vector Basis

But, we could choose any set of independent vectors as basis vectors, not just those along the x^0 - x^1 - x^2 - x^3 axes. They don't have to be unit vectors, but we will restrict our discussion to unit vectors only.

That is, we could have other basis vectors, in fact there are an infinite number of choices, other than (2). We can express these as

$$\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3 \quad \text{generally, as } \boldsymbol{\varepsilon}_r \quad r = 0, 1, 2, 3. \quad (4)$$

In terms of our coordinate axes basis (the Minkowski basis), we can express these as

$$\boldsymbol{\varepsilon}_0 = \varepsilon_0^\mu = (a_0, b_0, c_0, d_0) \quad \boldsymbol{\varepsilon}_1 = \varepsilon_1^\mu = (a_1, b_1, c_1, d_1) \quad \boldsymbol{\varepsilon}_2 = \varepsilon_2^\mu = (a_2, b_2, c_2, d_2) \quad \boldsymbol{\varepsilon}_3 = \varepsilon_3^\mu = (a_3, b_3, c_3, d_3), \quad (5)$$

which is parallel to (2).

Note that r represents the particular basis vector, and μ the components of that basis vector in the Minkowski spacetime coordinate system.

We can represent the same Z vector field of (3) in this different basis as

$$\begin{aligned} Z^\mu &= Z_r \boldsymbol{\varepsilon}_r = Z_{r=0} \boldsymbol{\varepsilon}_0 + Z_{r=1} \boldsymbol{\varepsilon}_1 + Z_{r=2} \boldsymbol{\varepsilon}_2 + Z_{r=3} \boldsymbol{\varepsilon}_3 = Z_{r=0} \varepsilon_0^\mu + Z_{r=1} \varepsilon_1^\mu + Z_{r=2} \varepsilon_2^\mu + Z_{r=3} \varepsilon_3^\mu \\ &= Z_{r=0} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} + Z_{r=1} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + Z_{r=2} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} + Z_{r=3} \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} = \begin{bmatrix} Z_{r=0} a_0 + Z_{r=1} a_1 + Z_{r=2} a_2 + Z_{r=3} a_3 \\ Z_{r=0} b_0 + Z_{r=1} b_1 + Z_{r=2} b_2 + Z_{r=3} b_3 \\ Z_{r=0} c_0 + Z_{r=1} c_1 + Z_{r=2} c_2 + Z_{r=3} c_3 \\ Z_{r=0} d_0 + Z_{r=1} d_1 + Z_{r=2} d_2 + Z_{r=3} d_3 \end{bmatrix} = \begin{bmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{bmatrix}. \end{aligned} \quad (6)$$

(6) and (3) are the exact same vector, just expressed in terms of different bases. For example, Z^1 is the amount of the Z field along the x^1 axis, i.e., aligned with the \mathbf{i}_1 basis vector. $Z_{r=1}$ is the amount of the Z field in the $\boldsymbol{\varepsilon}_1$ direction (which is generally not the same as the \mathbf{i}_1 vector).

As examples, Z^1 has generally four different components in the \mathbf{e}_r basis, each component aligned along one of the four \mathbf{e}_r basis vectors, though it only has one component (Z^1) in the Minkowski \mathbf{i}_s basis ($s = 0,1,2,3$). $Z_{r=1}$ is the amount of the Z vector aligned with the \mathbf{e}_1 vector, so only has one component ($Z_{r=1}$) in that basis, but it generally has four components in the Minkowski basis ($Z_{r=1}a_1, Z_{r=1}b_1, Z_{r=1}c_1, Z_{r=1}d_1$).

Typically, in classical theory and QFT, the \mathbf{e}_r vectors are taken to align with (linear) polarization directions of the field being studied. In general, the polarization directions do not align with the \mathbf{i}_s directions of the x^0 - x^1 - x^2 - x^3 coordinate system, though they can.

2 Traditional Polarization Vector Basis

In analyzing the photon field in Vol. 1, pgs. 142-143, it was found convenient to align our x^0 - x^1 - x^2 - x^3 coordinate system such that the x^3 axis was aligned with field 3-momentum 3-vector \mathbf{k} . Since the polarizations were then in the x^1 and x^2 directions, things became very simplified for our polarization vectors \mathbf{e}_r . That is,

$$\varepsilon_0^\mu = (1,0,0,0) \quad \varepsilon_1^\mu = (0,1,0,0) \quad \varepsilon_2^\mu = (0,0,1,0) \quad \varepsilon_3^\mu = (0,0,0,1). \quad (7)$$

Note how this plays out for the case of a massive field, as in Appendix A of Vol. 2, Chap. 8, pg. 293, where

$$Z^\mu = Z_{r=0}\mathbf{e}_0 + Z_{r=1}\mathbf{e}_1 + Z_{r=2}\mathbf{e}_2 + Z_{r=3}\mathbf{e}_3 = Z_{r=0}\varepsilon_0^\mu + Z_{r=1}\varepsilon_1^\mu + Z_{r=2}\varepsilon_2^\mu + Z_{r=3}\varepsilon_3^\mu = \begin{bmatrix} Z_{r=0} \\ Z_{r=1} \\ Z_{r=2} \\ Z_{r=3} \end{bmatrix} = \begin{bmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{bmatrix}. \quad (8)$$

We align our coordinate system such that \mathbf{k} of the field Z is directed along the positive x^3 axis. So,

$$k^\mu = (\omega_{\mathbf{k}}, 0, 0, |\mathbf{k}|) \quad k_\mu = (\omega_{\mathbf{k}}, 0, 0, -|\mathbf{k}|) \quad \text{where} \quad \omega_{\mathbf{k}}^2 - \mathbf{k}^2 = m_Z^2 \quad (9)$$

For massive fields, we have the constraint

$$k_\mu Z^\mu = 0 \quad \text{repeat of (8-122) in Vol. 2.} \quad (10)$$

Using (8) and (9), we find (10) gives us

$$k_\mu Z^\mu = (\omega_{\mathbf{k}}, 0, 0, -|\mathbf{k}|) \begin{pmatrix} Z_{r=0} \\ Z_{r=1} \\ Z_{r=2} \\ Z_{r=3} \end{pmatrix} = \omega_{\mathbf{k}} Z_{r=0} - |\mathbf{k}| Z_{r=3} = 0 \quad \rightarrow \quad Z_{r=0} = \frac{|\mathbf{k}|}{\omega_{\mathbf{k}}} Z_{r=3} \neq 0 \quad \rightarrow \quad Z^0 = \frac{|\mathbf{k}|}{\omega_{\mathbf{k}}} Z^3 \neq 0. \quad (11)$$

For massless fields like the photon, $|\mathbf{k}| = \omega_{\mathbf{k}}$, and as we found for photons, $Z_{r=0} = Z_{r=3}$. That simplified a lot of things, which we won't get into here. (See Vol. 1, Chap. 5). However, for massive fields $|\mathbf{k}| \neq \omega_{\mathbf{k}}$ and we lose those simplifications.

3 A Different Polarization Basis

We would like to employ a polarization vector basis that does simplify things for massive fields. In particular, it will help significantly if we can use such a basis where $Z'_{r=0} = 0$. It turns out that the following choice will do the job (as we will see). We use primes to distinguish from the traditional choice (7).

$$\begin{aligned} \varepsilon_0'^\mu &= \frac{1}{m_z} (\omega_{\mathbf{k}}, 0, 0, |\mathbf{k}|) & \varepsilon_1'^\mu &= (0, 1, 0, 0) & \varepsilon_2'^\mu &= (0, 0, 1, 0) & \varepsilon_3'^\mu &= \frac{1}{m_z} (|\mathbf{k}|, 0, 0, \omega_{\mathbf{k}}) \\ \varepsilon_{0\mu}' &= \frac{1}{m_z} (\omega_{\mathbf{k}}, 0, 0, -|\mathbf{k}|) & \varepsilon_{1\mu}' &= (0, -1, 0, 0) & \varepsilon_{2\mu}' &= (0, 0, -1, 0) & \varepsilon_{3\mu}' &= \frac{1}{m_z} (|\mathbf{k}|, 0, 0, -\omega_{\mathbf{k}}) \end{aligned} \quad (12)$$

Problem: Show that each vector in (12) has unit length and that it is orthogonal to all the other vectors in (12).

With (12), we get

$$Z^\mu = Z'_{r=0}\epsilon'_0 + Z'_{r=1}\epsilon'_1 + Z'_{r=2}\epsilon'_2 + Z'_{r=3}\epsilon'_3 = Z'_{r=0}\epsilon'^{\mu}_0 + Z'_{r=1}\epsilon'^{\mu}_1 + Z'_{r=2}\epsilon'^{\mu}_2 + Z'_{r=3}\epsilon'^{\mu}_3 = \begin{bmatrix} Z'_{r=0}\frac{\omega_{\mathbf{k}}}{m_z} + Z'_{r=3}\frac{|\mathbf{k}|}{m_z} \\ Z'_{r=1} \\ Z'_{r=2} \\ Z'_{r=0}\frac{|\mathbf{k}|}{m_z} + Z'_{r=3}\frac{\omega_{\mathbf{k}}}{m_z} \end{bmatrix} = \begin{bmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{bmatrix}, \quad (13)$$

which is the same Z vector as in (8), just expressed (everywhere except after the last equal sign) in terms of different polarization basis vectors.

Now, align our coordinate axis such that the x^3 axis is in the \mathbf{k} direction, so we get (9). Then use (10) with (9) and (13).

$$0 = k_\mu Z^\mu = (\omega_{\mathbf{k}}, 0, 0, -|\mathbf{k}|) \begin{bmatrix} Z'_{r=0}\frac{\omega_{\mathbf{k}}}{m_z} + Z'_{r=3}\frac{|\mathbf{k}|}{m_z} \\ Z'_{r=1} \\ Z'_{r=2} \\ Z'_{r=0}\frac{|\mathbf{k}|}{m_z} + Z'_{r=3}\frac{\omega_{\mathbf{k}}}{m_z} \end{bmatrix} = Z'_{r=0}\frac{\omega_{\mathbf{k}}^2}{m_z} + Z'_{r=3}\frac{\omega_{\mathbf{k}}|\mathbf{k}|}{m_z} - Z'_{r=0}\frac{|\mathbf{k}|^2}{m_z} - Z'_{r=3}\frac{\omega_{\mathbf{k}}|\mathbf{k}|}{m_z} \quad (14)$$

$$= Z'_{r=0}\frac{\omega_{\mathbf{k}}^2 - |\mathbf{k}|^2}{m_z} = Z'_{r=0}m_z \rightarrow Z'_{r=0} = 0.$$

Hence, if $Z'_{r=0} = 0$, then (13) becomes

$$Z^\mu = Z'_{r=1}\epsilon'_1 + Z'_{r=2}\epsilon'_2 + Z'_{r=3}\epsilon'_3 = Z'_{r=1}\epsilon'^{\mu}_1 + Z'_{r=2}\epsilon'^{\mu}_2 + Z'_{r=3}\epsilon'^{\mu}_3 = \begin{bmatrix} Z'_{r=3}\frac{|\mathbf{k}|}{m_z} \\ Z'_{r=1} \\ Z'_{r=2} \\ Z'_{r=3}\frac{\omega_{\mathbf{k}}}{m_z} \end{bmatrix} = \begin{bmatrix} Z^0 \\ Z^1 \\ Z^2 \\ Z^3 \end{bmatrix}, \quad (15)$$

and any factors with ϵ'_0 drop out of any calculation with Z^μ . The constraint equation (10) in this basis forces the zeroth polarization component in this basis to be zero.

Thus, the constraint (10), with the part of (15) after the second equal sign, becomes

$$0 = k_\mu Z^\mu = Z'_{r=1}k_\mu\epsilon'^{\mu}_1 + Z'_{r=2}k_\mu\epsilon'^{\mu}_2 + Z'_{r=3}k_\mu\epsilon'^{\mu}_3. \quad (16)$$

Since the Z'_r ($r = 1, 2, 3$) in (16) are all independent, each term in (16) must independently go to zero. Thus, we have

$$k_\mu\epsilon'^{\mu}_r = 0 \quad \text{for each } r = 1, 2, 3 \quad (8-124) \text{ of Vol. 2.}$$